ON THE SOLUTION OF A SYSTEM OF LINEAR EQUATIONS.

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A general system of m linear equations in n unknowns may be denoted as follows:

$$T=\begin{cases} a_{11}\ x_1+a_{12}\ x_2+\ .\ \ .\ \ .\ +a_{1n}\ x_n+k_1=0, \\ a_{21}\ x_2+a_{22}\ x_3+\ .\ \ .\ \ .\ +a_{2n}\ x_n+k_2=0, \\ .\quad\quad .\quad\quad .\quad\quad .\quad\quad .\quad\quad . \\ .\quad\quad .\quad\quad .\quad\quad .\quad\quad .\quad\quad . \\ a_{m1}\ x_1+a_{m2}\ x_2+\ .\ \ .\ \ .\ +a_{mn}\ x_n+k_m=0. \end{cases}$$

In the study of the solutions of this system the following two matrices are of especial importance.

$$A=\begin{vmatrix} a_{11} & a_{12} & . & . & . & a_{1n} \\ a_{21} & a_{22} & . & . & . & a_{2n} \\ & . & & . & & . \\ & . & & . & & . \\ a_{m1} & a_{m2} & . & . & . & a_{mn} \end{vmatrix}, \qquad B=\begin{vmatrix} a_{11} & a_{12} & . & . & . & a_{1n} & k_1 \\ a_{21} & a_{22} & . & . & . & a_{2n} & k_2 \\ & . & & . & & . & \\ & . & & . & & . & \\ a_{m1} & a_{m2} & . & . & . & a_{mn} & k_m \end{vmatrix}.$$

Capelli exhibited the pedagogic advantage in employing, in the study of the system T, the concept of rank of these matrices, which are known, respectively, as the matrix and the augmented matrix of the system. The rank of a matrix is, according to Frobenius, the order of the largest non-vanishing determinant formed by elements of the matrix in order. The theorem proved by Capelli may be stated as follows: The necessary and sufficient condition that the system T is solvable is that the rank of the matrix of T is equal to the rank of its augmented matrix.* By solvable is meant that a set of n *finite* values may be assigned to the unknowns x_1, x_2, ..., x_n, so that each of the equations in T is satisfied.

When the system T is solvable it is also said to be consistent or compatible. If the system T is consistent and of rank r we may assign arbitrary values to at least one set of $n-r$ unknowns in this system so that after this is done it is possible to solve the resulting system. To each such $n-r$ arbitrary values there will correspond a single set of values for the remaining r unknowns, provided the $n-r$ unknowns to which arbitrary values were assigned were so selected that the rank of the matrix of the system is not diminished by omitting the coefficients of these $n-r$ unknowns from the matrix. As this interesting theory is so clearly presented in Bôcher's

*Capelli, *Rivista di Matematica,* Vol. 2 (1892), page 54. Capelli used the term characteristic instead of rank. The former of these terms is frequently employed in France and Italy. Cf. Pincherle, *Lezioni di algebra complementare,* 1909, page 92.

Introduction to Higher Algebra it does not appear necessary to enter into greater details here.

The main object of the present note is to consider the question: when can a given unknown in a consistent system of equations have only one value. To make this question perfectly clear we may consider the following system of three equations in three unknowns:

$$
\begin{aligned}
2x-y+2z&=8,\\
4x-2y-z&=-4,\\
6x-3y+z&=-4.
\end{aligned}
$$

It is evident that the rank of the matrix of this system is equal to the rank of the augmented matrix, each being 2. For every arbitrary value of x there is one and only one pair of values for y and z such that each of these three equations is satisfied. For instance, when $x=0$, y and z must have the value 0, 4 respectively; and when $x=1$ the values of y and z are 2, 4 respectively. Similarly, there is one and only one pair of values of x and z for every arbitrary value of y. On the contrary, z must always have the same value, namely, 4. We have therefore a system here in which the unknown z can have only one value although the system has an infinite number of solutions.

We are now in position to understand the following theorem:

The necessary and sufficient condition that a given unknown in a consistent system can have only one value is that the rank of the matrix of the system is diminished by omitting the coefficients of this unknown from this matrix.

That this condition is necessary follows directly from the general theory mentioned above; for, if the rank of this matrix were not diminished by omitting the given coefficients we could assign an arbitrary value to this unknown and solve the resulting system. Hence it remains only to prove that the given condition is also sufficient.

Suppose that the system T is consistent and that the rank of its matrix A is r, but that the rank of the matrix A' obtained by omitting the coefficients of x_α from A is less than r. The rank of A' must therefore be $r-1$ as the co-factor of at least one of these coefficients cannot vanish in a non-vanishing determinant of order r contained in A. As the matrix A' is of rank $r-1$ each of its rows can be expressed as a linear function of $r-1$ of these rows.* Hence we may replace the system T by a new system T' having the following form:

$$
\begin{aligned}
a_{1\alpha}x_\alpha + l_1 + k_1 &= 0,\\
a_{2\alpha}x_\alpha + l_2 + k_2 &= 0,\\
\cdot \quad \cdot \quad \cdot \quad &\cdot \quad \cdot
\end{aligned}
$$

*Cf. Bocher, *Introduction to Higher Algebra*, page 36.

$$
\begin{array}{l}
\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
a_{r-1a}\, x_a + l_{r-1} + k_{r-1} = 0, \\
a_{ra}\, x_a + \phi_0(l_1, l_2, \ldots, l_{r-1}) + k_r = 0, \\
\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
a_{ma}\, x_a + \phi_{m-r}(l_1, l_2, \ldots, l_{r-1}) + k_m = 0,
\end{array}
$$

where $\phi_0, \ldots, \phi_{m-2}$ are the linear functions of $l_1, l_2, \ldots, l_{r-1}$.

The system T' is consistent since T is consistent and it must be of rank r since T has this rank. Hence it results that one and only one set of values for x_a, $l_1, \ldots, l_{r-1}$ will satisfy the first r equations. That is, x_a can have only one value in system T'. It can therefore have only one value in system T and the theorem in question has been established. From this theorem it results that each unknown in system T has either only one value or it has an infinite number of values whenever this system is consistent. In particular, a system of linear equations has always an infinite number of distinct solutions whenever it has more than one solution.

From the above it is evident that the language as regards solvability of a system of linear equations becomes much more concise by means of the concept of rank. Although this concept is comparatively new in mathematics it is of such fundamental importance that it should occupy a more prominent place in the courses in advanced algebra. It should be remembered that a thing can only appear simple when we can see clearly through it. In particular, the theory of linear equations appears simple only after an exhaustive study by means of such a powerful instrument as the concept of rank.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

GEOMETRY.

363. Proposed by G. I. HOPKINS, Manchester, N. H.

Construct the triangle, having given, base, vertical angle, and difference between altitude and sum of the other two sides.

Solution by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

In what follows we assume that $AB=a=$the given base; $\angle ACB=\angle C=$given vertical angle; $p=$difference of altitude and sum of other sides.